# On the Appropriateness and Necessity of Proofs and Proving in Middle School Mathematics Classrooms

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**Abstract:** Proofs are central to mathematics as a discipline, yet outside of high school geometry classes proofs and proving are often absent from school mathematics classes. The absence of proofs is detrimental to later student success in mathematics. Research indicates that proofs and proving, especially informal proofs, are appropriate for middle grades students and improve student understanding and confidence in their knowledge. This article discusses the research and calls for the inclusion of proofs and proving as a primary component of middle grades mathematics instruction.

## Introduction

How do we know what we know is true? For many students, school mathematics consists largely of memorizing procedures and learning when to apply those procedures to word problems to get the correct answer. But how do we know that the standard algorithm for addition or subtraction or multiplication or division is valid? How do we know that we can "invert and multiply" when dividing fractions? How do we know that the Pythagorean Theorem is valid for all right triangles and not just the examples we have seen? For many students, and many adults, the answer is that it was in the textbook, and it has worked so far without fail when executed correctly. But this does not represent genuine belief or understanding, and it does not represent how the field of mathematics approaches knowledge or knowing.

Mathematics as a discipline is built on series of proofs and logical arguments that establish knowledge and understanding of quantity, space, change and motion. All conjectures and arguments are subject to rigorous proof. The National Council of Teachers of Mathematics includes "Reasoning and Proof" as one of the five major processes of math education. However, proofs and proving are often markedly absent in school mathematics outside of high school geometry courses. When proofs or proving do appear in school mathematics, they are often proofs by multiple examples or other methods that do not stand up to mathematical methods or logical scrutiny.

Informal proofs, especially operative proofs, are not uncommon in the early grades when students are learning the four operations with whole numbers. And students are often taught to solve problems in ways that include an informal proof, by performing multiplication with arrays or adding and subtracting along a number line, for example. These are not general proofs, which would not be developmentally appropriate, but include the proof for the specific answer to the problem in the procedure.

With the introduction of operations with fractions in the middle grades, and especially the introduction of multiplication and division of fractions, the subject matter of mathematics becomes more abstract. These operations with fractions are introduced just as students are rapidly improving their capacity for abstraction. Yet proofs, operative or otherwise, are often absent from the middle grades' curriculum. Proofs, both formal and informal, would not be inaccessible to most students especially when demonstrated on number lines or two-dimensional geometric representations. At the very moment that teachers should be able to start to teach more advanced methods of proving, such as generalized proofs of theorems, procedures, and methods, they often resort to teaching rote memorization of procedure.

The absence of proofs and proving in school mathematics leads to many problems. First, it creates a disconnect between mathematics as taught in school and mathematics as performed by mathematicians. Students who go on to pursue higher level study of mathematics, or any field that relies heavily on mathematics such as engineering, astronomy, etc., will encounter challenges when confronted with the need to include proofs and proving into their mathematical practice. Perhaps the most significant problem is that it makes mathematical knowledge and procedure into something that is simply delivered from a teacher directly to a student. Because proofs and proving are central to constructing new mathematical knowledge, students are unable to construct new mathematical knowledge or ideas without actively involving themselves in tasks that lead to proofs. When students question the validity of procedures, especially in the middle grades, teachers are often ill-prepared to discuss proof of procedures, beyond just showing more examples.

## Proofs and Proving in Mathematics

Proofs play a central role in mathematics as a discipline. The forms of proof may vary between times and cultures but are always present. The ancient Greek mathematicians produced rigorous deductive proofs starting from explicitly stated axioms based on observation of geometrical shapes, however this form of proof was limiting. As mathematics advanced, algebraic analysis became an important method of proving. Because most mathematics problems prior to the 19th century related to physical objects, the correctness of a solution or method was often proven by how well it reflected reality. In the 19th and 20th centuries, the axiomatic method of proving re-emerged, leading to many discoveries including non-Euclidean geometry. Computers introduced new ideas about proof, as mathematicians could use brute computing power to solve complex problems. This led to the emergence of reliable "probabilistic" proofs, as computers could work out far more complex probability models to a far greater degree of certainty. This has raised questions about the fallibility of proofs and changed the ways mathematicians engage with different types of proofs (Kleiner, 1991).

In an influential paper in *Philosophica Mathematica*, Yehuda Rav (1999) argues that proofs are the "bearers of mathematical knowledge" and the theorems, procedures, rules, etc., are mere summaries. If educators aim to prepare students to engage with mathematics as a discipline, it is necessary to prepare them to engage with the primary means of transmitting this knowledge, or proofs, just as educators teach historiography, literary analysis and scientific experimentation in social studies, language arts and science classes.

## Proofs and Proving in Middle School Mathematics

The National Council of Teachers of Mathematics (2021), in addition to calling for "reasoning and proof" also stress the need for "communication" to be a major component of mathematics education for all grade levels. If proofs are, as Rav (1999) argues, the "primary bearers of mathematical knowledge," then proofs and proving activity satisfy both components.

Many schools have traditionally only focused on proofs and proving in Euclidean geometry courses, usually taught at the high school level. However, many curricular standards recommend teaching proof in all areas of mathematics and at all levels. (Stylianides, 2007a) This raises questions both about what proofs look like at various grade levels and whether students are developmentally ready to engage with proofs and proving.

Theorists and researchers have identified and classified many forms of informal proof in mathematics classroom. Enactive or operative proofs involve a physical action or manipulation of physical objects to prove a mathematical idea and are regularly used in early primary grades. For example, using counters arranged in columns to prove whether a number is odd or even, and to eventually generalize student understanding of even and odd numbers (Whitman, 2009). Visual and graphic proofs involve the drawing of diagrams and figures to visualize mathematical statements. Examples include many proofs of the Pythagorean theorem and multiplication using arrays. Arithmetic and algebraic proofs consist of proof by calculations for both specific and general statements (Tall, 1998).

### *Can Middle Grade Students do Proofs?*

Many mathematics teachers express difficulty in teaching students' proofs. They offer many reasons, including the lack of logical maturity or the student's unawareness of the necessity of proof. (Balecheff, 2017) However, there is extensive research to suggest that not only are students as young as third grade capable of engaging with mathematical proofs, the act of proof and proving is well suited to their mode of thought and to the social construction of knowledge.

A five-year case study (Maher & Martino, 1996) of a single student assigned a combinatorics problem as a first grader was able to progressively improve her arguments over the years. By the fourth grade she was able to verbally explain how she knew she had discovered every possible combination. By fifth grade she was able to produce a written proof of the same. Other studies suggest that young students are capable of more general proving activities.

During an observation of a third-grade mathematics class conducted by Stylianides(2007a), students were asked to prove the conjecture that the addition of two odd numbers would also produce an even sum. Some students asserted that this was true because they tried "18 examples" and all of them conformed to the conjecture. Other students disagreed because there are infinitely many odd numbers so they could never test them all. She presented an example using 7+7 by drawing 2 sets of seven hashmarks and circling pairs. She showed that the single leftover hashmark from each set would combine to make two, an even number. The proof continued that since every odd number consists of a number of pairs plus one remaining hashmark, there would always be some number of pairs plus one extra hashmark each, therefore the sum of any two odd numbers would be even. Some students objected that it only proved that 7+7 was even. But other students responded that her explanation proved that any sum of odd numbers would be even. Not only does this show third graders engaging in rigorous proving activity, but several of them also even understood that empirical proof was insufficient and insisted upon general proof.

### *Implementing Proofs and Proving in a Middle Grades Classroom*

The challenges to making proofs and proving a central part of middle grades mathematics are myriad. Many teachers have a limited view of proofs, associating them only with formal deductive proofs. Many textbooks and curriculum guides either ignore proofs and proving or engage in poor practices such as empirical proofs that do not lead to strong understanding or genuine belief in students.

The largest obstacle in middle grades, however, is that most students have little experience with proofs and proving when they arrive in a teacher's classroom. This is a challenge but is not an insurmountable obstacle. It does, however, require the educator to be very intentional about including some form of proof in the introduction of any new concept or procedure, as well as for isolated questions or problems.

Some students only provisionally accept mathematical knowledge without proof. Other students prefer to simply learn a rote procedure. But the role of mathematical educators is to engage students with the discipline of mathematics, not to merely teach students to perform calculations that they do not genuinely understand.

### *Why Proofs and Proving are Important for Student Knowledge and Future Learning*

As some educators argue that proofs and proving are too difficult for middle grades students, others claim they are not necessary for students to achieve conceptual and procedural fluency. But extensive research suggests otherwise. One study found that students between 11- and 16-years old hold genuine beliefs about mathematical statements they have proven, but only provisionally accept empirical evidence as general proof (Porteous, 1990).

Some theorists of mathematics education caution against an emphasis on purely empirical means of explaining material, claiming it leads to a "prototype" model of understanding general mathematical concepts (Balecheff, 2017), or student acceptance of mathematical statements based in incomplete information, as opposed to a rigorous proof or argument.

## Conclusion

Educators within any content area seek to engage students with their discipline in ways that are consistent with the norms of that discipline. Science classes focus on the scientific process and scientific method. From the early grades students are taught to design experiments, carry them out and report their results. That is, they are taught to do science. ELA classes from the early middle grades engage in interpretation and criticism of literature. Social studies classes teach historiography and teach students how to engage with and evaluate primary and secondary sources.

Mathematics should strive for the same high standards, which means teaching students to engage with proofs and proving activities, because that is the work of mathematicians. There are those in society who argue that elementary and middle grades mathematics education should be purely applied and practical to everyday use and reject the idea that students in those grades need to learn to think like mathematicians. Not only does that argument degrade the discipline of mathematics to lesser than the other school disciplines, but such an approach deprives students of genuine knowledge and understanding. And a mathematics education that prepares students to engage in rigorous proof not only prepares them to engage with practical everyday problems but also prepares them to engage with novel problems whose solutions might benefit from a mathematical approach.

Proofs and proving have additional benefits for students in mathematics classes. Rigorous proof, especially when conducted by students themselves leads to both stronger understanding and confidence in their understanding (Porteous, 1990). It leads to less confusion when students advance to more complex mathematical ideas as they can build on existing knowledge in which they are confident. The process of proving is also a rigorous exercise in argumentation, which is a key element of every academic discipline, thus proofs and proving contributes to student's argumentation skills in other disciplines.

Proofs and proving activities should be included throughout middle grade math education, and all grades for that matter. No new concept or procedure should be introduced without some student engagement with proof of that concept or procedure. It leads to better understanding and teaches students to engage with mathematics as a discipline that seeks to understand the world, not merely as a tool for performing calculations.

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## About the Author

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