

Proof and Discourse in Mathematics Teaching for Competency

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Abstract: Recent changes to the standards for mathematics education have shifted focus towards the importance of proof, reasoning, explanation, all of which are essential components of understanding, recording, communicating, and doing mathematics. Yet research continues to show that American students (high school and beyond) struggle to understand the idea of proof in mathematics. The research demonstrates the importance of teaching mathematics through the lens of proof, and how meaningful mathematics discourse can be the catalyst for learning mathematical proof. Evidence shows that students become more capable learners and doers of mathematics when they understand and can construct mathematical proofs. Additionally, research shows that through engaging in mathematical discourse, students can learn the process of developing mathematical arguments more effectively.

Introduction

Historically, proof has been confined to a small corner in high school mathematics, only showing up regularly in geometry classes. Proof, however, is considered an essential component of understanding, recording, communicating, and doing mathematics (Knuth, 2002; Martin, McCrone, Bower, & Dindyal, 2005). Unfortunately, research also has shown that American students at the high school level and beyond struggle to understand proofs in mathematics (Martin, McCrone, Bower, & Dindyal, 2005; Miyazaki, Fujita, & Jones, 2017). With American students ranked just 38th globally in 2015 for mathematics achievement, nine spots below the OECD average (DeSilver, 2017), it is time for us to take a serious look as to when, where, and how we teach proof in mathematics.

One method which has been proven to develop students' abilities to complete and understand proof in mathematics is engaging students in mathematical discourse. Stylianou & Blanton (2011) link the ability to argue mathematically to well-organized classroom discourse. Such mathematical argumentation lays the groundwork for mathematical proof, because through argument students explain, justify, and rationalize their answers to questions. Therefore it is essential that teachers facilitate meaningful mathematical discourse in their classrooms. Far too often though, students in the United States are not provided with opportunities to engage in meaningful mathematics classroom discourse (Ryve, Nilsson, Pettersson, 2013).

This paper highlights the importance of teaching mathematics through proof, and explores how fostering meaningful mathematics discourse can enable students to gain competency with proofs and proving techniques. Additionally, it will argue that it is important that proof and discourse take place at each grade level and in each mathematical domain (e.g. algebra, geometry, trigonometry).

Proof

When

Jerome Bruner, a renowned 20th century educational psychologist, argued that any subject can be taught with a degree of rigor to any student at any stage of development. This notion underlies the idea of the “spiral curriculum,” in which students revisit a concept at various points throughout their schooling, with increasing complexity; through this spiraling new learning is connected to old (Gibbs, 2014). While many concepts in math are generally taught following a spiral curriculum (for example, functions are introduced early, then revisited), proofs have somehow been left out of the early grades, and even in secondary school, aside from geometry. As Hung-Hsi Wu (cited in Knuth, 2002) states, one “glaring defect in present-day mathematics education in high school” is “the fact that outside geometry, there are essentially no proofs.” According to Wu, this “presents a totally falsified picture of mathematics itself” (p. 228). With so few experiences with proof, it is no surprise that many secondary mathematics students find the study of proof difficult (Knuth, 2002).

The struggles students experience when studying proof need not be so great. Applying Bruner’s idea of “readiness to learn” along with appropriate scaffolding, even the youngest students can be asked to provide simple justification and explanations for their mathematical work. Since proof is a process of arguing, questioning statements, and using evidence appropriately, when young students are challenged to justify and defend their work they will be more prepared for the rigor and variation of the proofs that can be expected at the secondary, and even post-secondary, level (Stylianou & Blanton, 2011). The techniques used in younger grades may not appear to be ‘proofs,’ but any time students are challenged to explain their work, communicate their ideas, or critique their misconceptions, they are developing the reasoning skills that will be applied to more rigorous proofs later on.

Where

Just as proof, or some form of reasoning or explanation, should be required of mathematics students at all grade levels, the same requirements should be in place across the various branches of mathematics (Gonzalez & Hinthorn, 2003). One explanation for why proof has historically been contained to geometry is that the foundation for geometric proof is given in geometry classes, including postulates, axioms, definitions, and theorems. This is in contrast to the traditional Algebra class, for example, where the foundational properties -- such as field properties, properties of equality, and properties of real numbers -- are not formally given or explored with students, (Gonzalez & Hinthorn, 2003). If we equip our students with the rigorous tools of the discipline of mathematics, not only in geometry, but all areas of math, the students can then be given rich mathematical tasks that require them to justify their solutions and gain experience with mathematical proof, in any branch of mathematics at any grade level (Knuth & Elliott, 1998). Proof can no longer be contained to just geometry, because it has been upgraded to be its own standard in recent curriculum legislation, rather than being linked to a specific content domain

(Knuth, 2002; Council of Chief State School Officers, 2010). For this reason it is paramount that we provide our students opportunities to engage in rich, thought provoking proofs throughout each subject in their mathematics education.

How

This section attempts to answer the question of how teachers should select proofs which will provide meaningful understanding for students across grade levels and mathematics subject areas. The research presented will explain the nature of the proofs, or proving techniques, teachers can employ in their classrooms that will most enable their students to gain significant conceptual understanding, regardless of grade or mathematical domain (i.e. pre-algebra, geometry, calculus, etc.).

Knuth (2002), mathematics education researcher, suggests an approach that attempts to solve, or at least to mitigate, the problems surrounding teaching and learning proofs. In his work, Knuth examines the pedagogical function of proofs and their explanatory nature. He states:

Mathematicians recognize that the primary role of proof in mathematics is to establish the truth of a result; yet perhaps more important, particularly from an educational perspective, is their recognition of its role in fostering understanding of the underlying mathematical concepts (p. 478).

This notion of ‘explanatory proofs’ is what Knuth and other mathematics education experts credit as the most valuable learning and level of understanding of mathematics (Weber, 2003). As Hanna (as cited in Knuth, 2002) explains:

True understanding demands that students see why it is the case, and furthermore why it must always be the case, and this understanding is best engendered by explanatory proofs.

A proof that proves:

Prove: The sum of the first n positive integers is $n(n + 1)/2$.

For $n = 1$ it is true, since $1 = 1(1 + 1)/2$

Assume it is true for some arbitrary k , that is, $S(k) = k(k + 1)/2$. Then consider:

$$\begin{aligned} S(k + 1) &= S(k) + (k + 1) \\ &= k(k + 1)/2 + k + 1 \\ &= (k + 1)(k + 2)/2 \end{aligned}$$

Therefore the statement is true for $k + 1$ if it is true for k . By induction, the statement is true for all n (p. 8).

Hence, as teachers, we must enable our students to be exposed to a variety of proofs and proving techniques that allow for an explanatory element in order

for significant understanding of the underlying concepts to take place. Above is a proof which illustrates the nature of explanatory proofs.

This is one example of a proof with a variety of ways to explain and justify the underlying mathematical principle. It is then the responsibility of the teacher to select those problems in which their students will be exposed to explanatory arguments and counterexamples. This challenge need not be so great. Educational researcher Russo (2018) identifies 3 principles for developing explanatory ‘proof-type’ problems. His work focused on teaching proof in primary grades and his principles can be applied at any grade level, for any branch of mathematics.

For his first principle, Russo (2018) states that, “The problem should be worded as a statement, followed by a follow up question. “True or False? Prove it”” (p. 35). Presenting the problem as a conjecture which can be proven or falsified is the first step in forcing students to take a side and begin gathering evidence to support their claim. This method is in stark contrast to traditional methods of posing mathematics questions (Russo, 2018). The second principle states, that the “mathematical knowledge required to engage productively with the problem is accessible to most students beforehand” (p. 35). The nature of any proof-type question is cognitively challenging, especially for students with little proof-making experience. Since proof-like arguments often require a synthetization of various mathematical ideas and principles, it is important to ask questions that the students have the tools to answer. Finally, principal three states that an “important mathematical idea should lie at the heart of the problem” (p. 35). This is the stage in which teachers consider how to take concepts their students know and work them into a problem that will help them transition to understanding the principles which make the property or principal valid (Russo, 2018). This can be accomplished in a variety of manners, such as discovery-based problems, visual representations of arguments or principals, and in group presentations. Following these three principals, and keeping in mind the importance of having an explanatory nature, teachers can create meaningful opportunities for students of all ages, in any branch of mathematics, to engage with proof and proving techniques.

Once the material is selected, the next question is, “How should it be taught?” Insight as to how teachers can orchestrate lessons which provide their students opportunities to engage in a variety of meaningful proof investigations will be provided in the next section of this paper.

Discourse

When

The importance of meaningful discussion in mathematics has been clear for some time now, with the overwhelming majority of the research demonstrating the benefits of discourse in mathematics education, especially as it pertains to understanding and constructing proofs (Knuth, 2002; Mueller, Yankelewitz, & Maher, 2012; Russo, 2018; Ryve, Nilsson, & Pettersson, 2013; Stylianou & Blanton, 2011; Weber, 2003). Furthermore, the importance of allowing young, primary school students opportunities to engage in meaningful mathematics discourse is also well-supported by

research (Mueller, Yankelewitz, & Maher, 2012; Russo, 2018; Ryve, Nilsson, & Pettersson, 2013). Therefore, it is important that students be given opportunities early on in their education to discuss the mathematical ideas they are presented with in order for them to begin to formulate an idea for what mathematical argumentation is and is not. This knowledge of argumentation, born from discourse with classmates, teachers, and even themselves, is the groundwork for understanding and constructing mathematical proof (Stylianou & Blanton, 2011). This is not to say, however, that one should expect the same level of complexity in conversation, reasoning, and argumentation from a third grade class as one would a ninth grade class, but rather that teachers must consider students' readiness to learn and tailor discussions and activities to meet their students at an appropriate level of rigor and difficulty (Russo, 2018; Stylianou & Blanton, 2011).

Where

Again, as with proof, discourse should be taking place in a variety of mathematical courses. Mueller, Yankelewitz, and Maher (2012) studied students engaging in discourse related to a variety of mathematical applications (including geometry, probability, and arithmetic) and concluded that “the reasoning that emerged from their shared discourse was useful in the formation of each student’s individual understanding” (p. 384). Regardless of the specific mathematical subject being taught, students should be provided with opportunities to collaborate, share, discuss, and engage in all other manners of discourse in order to develop a deeper personal understanding of the content which will be required to complete mathematical proofs.

How

We now seek to answer the more challenging question of how teachers can utilize discourse to assist their students in understanding and constructing explanatory proofs across grade levels and mathematical domains. A problem many teachers encounter when teaching proofs is students’ instincts to develop empirical arguments, or to rely on a small set of examples to verify or disprove a conjecture (Stylianou & Blanton, 2011). Of course, sound mathematical proofs need to be generalized, to hold true for infinite iterations. So how do teachers assist their students in transitioning from empirical arguments to more general, explanatory proofs? For simplicity’s sake, imagine a lesson in which students work in cooperative groups to complete an explanatory proof (the manner of instruction doesn’t matter). In these cases, the use of “transactive prompts,” a practice in which the instructor repeatedly asks for clarifications, explanations, criticisms, and elaborations has been shown to assist students in constructing more comprehensive and valid arguments (Stylianou & Blanton, 2011). This practice can be employed regardless of the manner of instruction and will assist in developing deeper content knowledge if appropriate proof-type problems are assigned. As Stylianou & Blanton (2011) explain, “This type of discourse then became the avenue by which students learned to construct proofs” (p. 144).

The role of the teacher as a facilitator and provider of structure is also important in the process of mathematical discourse. Teachers can support the direc-

tion of students' thinking, organizing involved summaries of discussion, pacing the classroom conversation, and redirecting the focus of students' ideas and thinking as necessary (Stylianou & Blanton 2011).

Finally, it is important that the teacher maintain accountability amongst their students. The idea of fostering meaningful discourse is not to simply present ideas, but rather that the class should engage in scrutinizing and critiquing the reasoning and logic presented in arguments. As explained by Stylianou & Blanton (2011), students:

propose ideas that, once approved by the group, will become part of a common culture and, subsequently, building stones of future proofs. Thus, it is the group's responsibility to vet these proposed ideas, examine them, and take full responsibility for accepting them (p. 144).

The need for accountability lies with the whole group. It is the teacher's responsibility to maintain and challenge the accountability of the class.

Conclusion

Teachers must understand the importance of teaching proof from the early grades on, across the various branches of mathematics, in a manner which attempts to explain a mathematical idea. Additionally, teachers should be aware of the benefits that utilizing discourse strategies has on learning to construct and understand mathematical proof. Proof-type questions should be of an explanatory nature and offer students a variety of ways to express their mathematical ideas. When engaging in classroom discourse, the instructor should be utilizing transactive prompts, providing structure for and facilitating conversations, and maintaining the accountability of the class in regards to their arguments.

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About the Author

Bratche Eldred graduated from BGSU with a bachelor's degree in AYA mathematics in 2015. He is currently pursuing his Master of Arts and Education in Mathematics. A mathematics instructor at Jesup W. Scott H.S. in Toledo for the last three years, Bratche's goal is to provide his inner-city students opportunities to learn advanced mathematics for college credit.